# Compressibility Effects on Oscillating Rotor Blades in Hovering Flight

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**A general theory is developed for determining the aerodynamic forces acting on a multibladed rotor. The usual two-dimensional mathematical model is adopted in which a typical section of the reference rotor blade and its helical wake is replaced by an airfoil with a straight wake and a system of regularly spaced infinite wakes below including those from other blades. Aerodynamic coefficients are derived for particular values of Mach number, frequency, phase difference between blade motions, and wake spacing for a single blade and a two-bladed rotor. At the higher speeds the coefficients vary considerably with Mach number but variations with blade phasing are similar to those found for low rotational speeds. The mathematical model used yields the result that, for given circulation around the reference blade section, the downwash induced by the infinite system of wakes below it is independent of Mach number.**

## Nomenclature

| | |
|---|---|
| $l$ | = half-chord |
| $z\ (= lz'e^{ipt})$ | = displacement downward at mid-chord |
| $\alpha\ (= \alpha' e^{ipt})$ | = angular displacement |
| $U,M$ | = wind-speed and Mach number |
| $x = lX,\ z = lZ/\beta$ | = nondimensional coordinates |
| $\omega(= pl/U)$ | = frequency parameter |
| $\nu = \omega/\beta^2, \kappa = M\nu,$ $\lambda = M^2\nu, \beta = (1 - M^2)^{1/2}$ | = other parameters |
| $w(= w'e^{ipt})$ | = downwash distribution |
| $\phi(= l\Phi e^{i(\lambda X + \omega T)}$ | = velocity potential |
| $lk(= \phi_a - \phi_b)$ | = discontinuity in velocity potential across vortex sheet |
| $K(= \Phi_a - \Phi_b)$ | = discontinuity in $\Phi$ |
| $Q$ | = number of blades in rotor |
| $q$ | = blade number ($q = 0, 1, 2, Q - 1$) |
| $\Omega$ | = angular velocity of blade rotation |
| $\epsilon(= p/\Omega)$ | = frequency ratio |
| $\psi_0(= 2\pi\epsilon)$ | = phase lag per rotation |
| $\psi_q$ | = phase lead of $q$th blade |
| $dl$ | = spacing of vortex sheets |
| $X(= -\cos\theta)$ | = for points on the reference airfoil |

## 1. Introduction

As part of a general program of research on unsteady aerodynamic problems sponsored by the Department of Defense, under a Project Themis contract, a study has been made of compressibility effects on the aerodynamic forces acting on oscillating rotor blades in hover. The corresponding problem for incompressible flow has been considered previously by a number of authors,[1-4] but it is believed that similar studies have not been made of the compressible flow problem. This has now become of major importance with the advent of helicopters operating with high rotor blade tip speeds.

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In the present paper, a method is developed for calculating the aerodynamic forces on oscillating rotor blades with tip speeds up to $M = 0.9$ for a range of frequency parameter values of practical interest. The theory can be applied to a multibladed rotor.

The usual two-dimensional mathematical model of the flow is adopted in which a section of the reference rotor blade and its helical wake is replaced by an airfoil in straight flight with a system of infinite wakes under it as described in the next section. As well as being influenced by its own wake system, the reference blade will be affected by the wakes of the other blades of the rotor which are assumed to be oscillating with the same frequency and amplitude but not necessarily in phase.

This type of model has been used successfully by many investigators who were able to obtain satisfactory agreement between theory and experiment in their respective studies of rotor instabilities. The importance of tip effects, neglected in the two-dimensional model, has been considered by Greidanus, Timman, and Zaat.[4] They concluded that the two-dimensional model representation of the flow is adequate for airload estimation and flutter investigations.

On the basis of the available evidence, it can therefore be assumed that the model would give a reasonable first approximation to the airload coefficients for Mach numbers less than $M = 0.8$. At Mach numbers in this region and above, severe shockwave boundary-layer interaction effects would be present and linearized theory would then be inadequate. When the influence of the tip vortices is also taken into account the flow over the outer portion of the blade is likely to be far from two-dimensional. Though the present theory is limited in that it cannot deal with such effects, it does provide a basis for further study of such problems.

## 2. Mathematical Model

In a paper by Loewy[1] the two-dimensional model representation of a multibladed rotor is discussed in detail. A similar model is used in the present paper in which the system of helical wakes from the blades is represented by a two-dimensional array of regularly spaced vortex sheets. Let us suppose that the rotor has $Q$ blades and that a particular blade is denoted by $q$, where $q$ takes the value of $0, 1, 2, 3, \ldots, Q - 1$, in turn and $q = 0$ represents the reference blade. To determine the flow over a particular section of the reference blade at radius $r$, it is assumed that the blade section can be replaced by an airfoil of the same chord moving with velocity $U(\equiv r\Omega)$ in a straight line and oscillating in the same

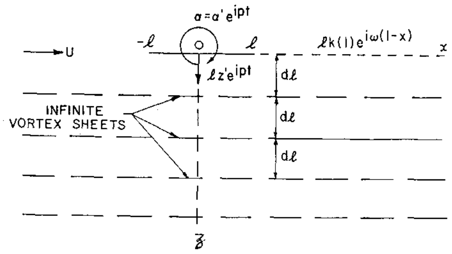


**Fig. 1 Mathematical model of airfoil and vortex sheets.**

manner. The helical wakes are then replaced by a semi-infinite wake due to the airfoil itself together with a system of infinite wakes as shown in Fig. 1. If the rotation is such that blade $q - 1$ follows blade $q$, the semi-infinite wake of the airfoil representing the reference blade section will be followed by the infinite wakes arising from the blades 1, 2, 3, . . ., $Q - 1$ in that order in the first revolution. The pattern repeats itself for every revolution thereafter except that the wake from the airfoil, $q = 0$, also extends from $-\infty$ to $+\infty$.

The spacing between the wakes is determined by the inflow and the rotational speed. Let us suppose that consecutive wakes are separated by a distance $dl$ where $2l$ is the airfoil chord. For a low inflow, $d$ will be small and when this is the case, the two-dimensional model of the flow used is believed to give reasonable results.

All the blades of the rotor are assumed to be oscillating with the same amplitude and frequency but not necessarily in phase. If the section of the reference blade represented by the airfoil is describing plunging oscillations defined by $z = lz = lz' \exp(ipt)$, and pitching oscillations $\alpha = \alpha' \exp(ipt)$ about the midchord point, taken as origin of the coordinate system, the displacement $\zeta$ of any point $P$ at a distance $x(= lX)$ from midchord is given by

$$\zeta = l(z + X\alpha) = l(z' + X\alpha')e^{ipt} \tag{1}$$

where $p$ is the frequency in rad/sec. Then, if $\phi$ is the corresponding velocity potential of the disturbed flow, the condition for tangential flow yields

$$\partial\phi/\partial z = w = d\zeta/dt = (\partial/\partial t + U\partial/\partial x)\zeta \tag{2}$$

From (1) and (2) it readily follows that the downwash $w(X)$ is given by

$$w(X) = U[i\omega z' + \alpha'(1 + i\omega X)]e^{ipt} = w'e^{ipt} \tag{3}$$

where $\omega = pl/U$ is the frequency parameter. The problem is then reduced to one of finding a vorticity distribution which will induce a downwash distribution $w(X)$ as defined by (3).

## 3. Basic Theory

The two-dimensional unsteady airfoil theory for compressible flow has been fully developed and in the present analysis use will be made of well-known results. It can readily be deduced from Euler's equations of motion that the velocity potential $\phi$ of the disturbed flow satisfies the equation

$$d^2\phi/dt^2 = U_s^2(\partial^2\phi/\partial x^2 + \partial^2\phi/\partial z^2) \tag{4}$$

where $U_s$ denotes the speed of sound.

Next, let the coordinates be changed and write

$$x = lX, z = lZ/\beta, t = lT/U \tag{5}$$

where $\beta = (1 - M^2)^{1/2}$ and $M$ is the Mach number of the local flow at the reference section of the blade under consideration. The velocity potential of the disturbed flow can then be expressed in the form

$$\phi = l\Phi e^{i(\lambda X + \omega T)} \tag{6}$$

where $\lambda = M^2\omega/(1 - M^2)$. From (4) and (6) it may then be deduced that the modified potential $\Phi$ satisfies the wave equation;

$$\partial^2\Phi/\partial X^2 + \partial^2\Phi/\partial Z^2 + \kappa^2\Phi = 0 \tag{7}$$

where $\kappa = M\omega/(1 - M^2)$.

If $\phi_a$ and $\phi_b$ denote the values of $\phi$ immediately above and below the vortex sheet representing the airfoil and its wake, it follows that

$$\phi_a - \phi_b = lk(X) = lK(X)e^{i(\lambda X + \omega T)} \tag{8}$$

where $K(X) = \Phi_a - \Phi_b$. By the use of Euler's equations, it can then be deduced that the lift distribution $\bar{l}(x)$ over the airfoil is given by

$$\begin{aligned} \bar{l}(x) &= \rho U(i\omega k + \partial k/\partial x) \\ &= \rho U(i\nu K + \partial K/\partial X)e^{i(\lambda X + \omega T)} \end{aligned} \tag{9}$$

where $\nu = \omega/(1 - M^2)$. If a function $\Gamma(X)$ is now introduced such that

$$\Gamma(X) = i\nu K + \partial K/\partial X \tag{10}$$

it can readily be shown that

$$K(X) = e^{-i\nu X}\int_{-1}^{x} \Gamma e^{i\nu X}\, dX \tag{11}$$

When $X \geq 1$ in the wake, $\bar{l}(x)$ is zero and hence $\Gamma(X)$ must vanish. It then follows that in the wake

$$K(X) = e^{-i\nu X}\int_{-1}^{1} \Gamma e^{i\nu X}\, dX = K(1)e^{i\nu(1-X)} \tag{12}$$

where $K(1)$ denotes the value of $K(X)$ at the trailing edge. From (6), by differentiation, it also follows that the modified downwash amplitude

$$W = \partial\Phi/\partial Z = we^{-i(\lambda X + \omega T)}/\beta = w'(X)e^{-i\lambda X}/\beta \tag{13}$$

where $w'$ is defined by (3).

For the case of a single airfoil oscillating in compressible flow, it has been shown that the solution of the problem can be derived from the integral equation[5]

$$2\pi W(X_1,Z_1) = -\int_{-1}^{\infty} K(X)\, \frac{\partial^2}{\partial Z_1^2}\left[\frac{\pi i}{2}\, H_0^{(2)}(\kappa r_1)\right]dX \tag{14}$$

where $r_1 = [(X - X_1)^2 + Z_1^2]^{1/2}$ and $Z_1 \rightarrow 0$. On the airfoil, $W$ is given by (13) and the problem is then reduced to one of finding a doublet distribution $K(X)$ which satisfies (14) and the condition, $\Gamma = 0$, in the wake. The lift distribution is given by (9) and, when $K(X)$ is known, the aerodynamic forces acting on the airfoil can be determined.

However, for the problem considered in this paper the influence of the infinite system of wakes under the airfoil must be taken into account and Eq. (14) has to be modified accordingly to allow for the effects of the wakes of all the blades. The doublet distribution corresponding to the reference blade of a multibladed rotor after $n$ revolutions is denoted by

$$\begin{aligned} K_{0n}(X) &= K(1)\exp[i\nu(1 - X) - 2\pi ni\epsilon] \\ &= K_{00}(X_1)\exp[-i\nu(X - X_1) - 2\pi ni\epsilon] \end{aligned} \tag{15}$$

where $\epsilon \equiv p/\Omega$, $\Omega$ being the angular velocity of rotation. Similarly, if the motion of the blade $q$ has a phase lead of $\psi_q$ relative to the reference airfoil, the associated doublet wake distribution after $n$ revolutions is given by

$$K_{qn}(X) = K_{00}(X_1)\exp[-i\nu(X - X_1) + i\psi_q - 2\pi i\epsilon(n + q/Q)] \tag{16}$$

and this wake will be at a distance $(nQ + q)D$ from the reference airfoil with $D \equiv d\beta$. If the contributions of all the wakes to the modified downwash is denoted by $W_i(X_i)$, Eq. (14) is replaced by

$$2\pi[W(X_1) - W_i(X_i)] = -\int_{-1}^{\infty} K(X)\frac{\partial^2}{\partial Z_1^2}\left[\frac{\pi i}{2}H_0{}^2(\kappa r_1)\right]dX \qquad (17)$$

where

$$2\pi W_i(X_1) = -K_{00}(X_1)\sum_{n=1}^{\infty}\exp(-2\pi n i\epsilon)P_o - K_{00}(X_1)\sum_{n=0}^{\infty}\sum_{q=1}^{Q-1}\exp\left[i\psi_q - \frac{2\pi i\epsilon}{Q}(nQ+q)\right]P_q \qquad (18)$$

$$P_0 = \frac{\pi i}{2}\int_{-\infty}^{\infty} e^{-i\nu\xi}\frac{\partial^2}{\partial Z_1^2}H_0{}^{(2)}(\kappa r_2)d\xi$$

$$P_q = \frac{\pi i}{2}\int_{-\infty}^{\infty} e^{-i\nu\xi}\frac{\partial^2}{\partial Z_1^2}H_0{}^{(2)}(\kappa r_3)d\xi$$

$$r_2 = [\xi^2 + (nQD - Z_1)^2]^{1/2},\ r_3 = [\xi^2 + [(nQ+q)D - Z_1]^2]^{1/2}$$

and $\xi = X - X_1$. This expression can be simplified by making use of the following known relation:[6]

$$\int_{-\infty}^{\infty} e^{-i\nu\xi}H_0{}^{(2)}[\kappa(\xi^2+\eta^2)^{1/2}]d\xi = \frac{2ie^{-\eta\nu\beta}}{\nu\beta} \qquad (19)$$

where $\kappa = M\nu$. It can then be deduced by direct differentiation or by integration by parts that

$$\int_{-\infty}^{\infty} e^{-i\nu\xi}\frac{\partial^2}{\partial\eta^2}H_0{}^{(2)}[\kappa(\xi^2+\eta^2)^{1/2}]d\xi = 2i\nu\beta e^{-\eta\nu\beta} \qquad (20)$$

With the use of (20), Eq. (18) yields, when $Z_1 \to 0$,

$$2\pi W_i(X_1) = \pi\nu\beta K_{00}(X_1)\sum_{n=1}^{\infty}e^{-n\gamma} + \pi\nu\beta K_{00}(X_1)\sum_{q=1}^{Q-1}\exp\left[i\psi_q - \frac{2\pi q i\epsilon}{Q}\right]\sum_{n=0}^{\infty}e^{-n\gamma - qD\nu\beta} \qquad (21)$$

where $\gamma = 2\pi i\epsilon + Q\nu D\beta = 2\pi i\epsilon + Q\omega d$. After summing the series in (21), we derive the formula

$$2\pi W_i(X_1) = (\pi/QD)K_{00}(X_1)\,F(Q) \qquad (22)$$

where

$$F(Q) = \frac{Q\nu D\beta}{e^\gamma - 1}\left[1 + \sum_{q=1}^{Q-1}\exp\left\{i\psi_q + \frac{Q-q}{Q}\gamma\right\}\right]$$
$$= \frac{Q\omega d}{e^\gamma - 1}\left[1 + \sum_{q=1}^{Q-1}\exp\left\{i\psi_q + \frac{Q-q}{Q}\gamma\right\}\right] \qquad (23)$$

provided $|e^{-\gamma}| < 1$. It should be noted that $F(Q)$ is independent of Mach number and that since

$$K_{00}(X_1) = K(1)e^{i\nu(1-X_1)}$$

Eqs. (13) and (22) yield the following formula for the actual downwash $w_i(X_1)$ due to the infinite system of wakes, namely

$$2\pi w_i(X_1) = \frac{\pi k(1)e^{i\omega(1-X_1)}}{Qd}F(Q) \qquad (24)$$

Since the expression on the right is independent of Mach number we arrive at the important result that for a given circulation, $lk(1)$, the downwash due to the infinite system of wakes is the same for compressible and incompressible flow.

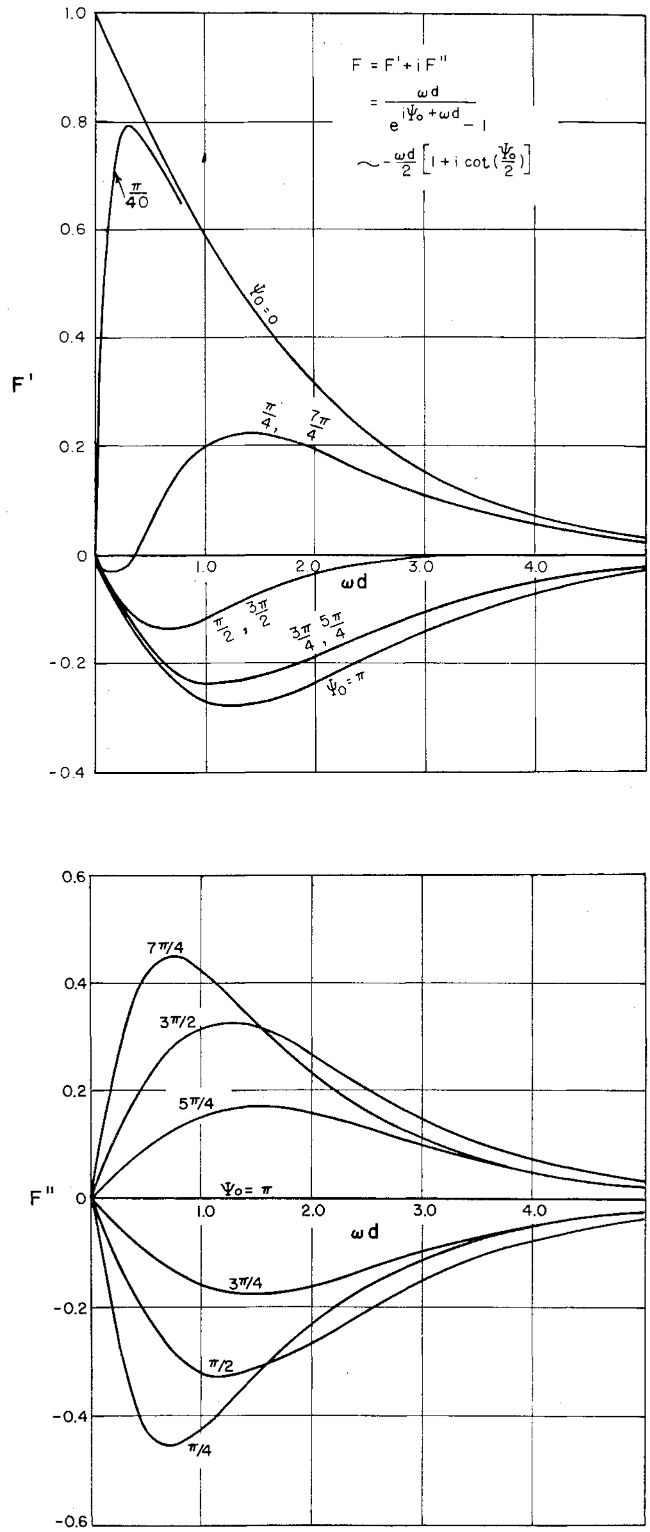


**Fig. 2 Variation of $F$ with $\omega d$ and phase lag for a single blade.**

The real and imaginary parts of $F$ for a single blade are shown plotted in Figs. 2a and 2b below.

It has been pointed out by Loewy[1] that the downwash due to the infinite wakes of a $Q$ bladed system can, under special conditions, correspond to that for an equivalent single bladed rotor. For a single blade

$$2\pi w_i(X_1) = \pi\omega k(1)e^{i\omega(1-X_1)}/(e^{2\pi i\epsilon + \omega d} - 1) \qquad (25)$$

and for a multibladed rotor, when all the $\psi_q$'s are zero,

$$2\pi w_i(X_1) = \pi\omega k(1)e^{i\omega(1-X_1)}\Big/\left(\exp\left[\frac{2\pi i\epsilon}{Q} + \omega d\right] - 1\right) \quad (26)$$

Hence, the equivalent parameters $d_e$ and $\epsilon_e$ are $d_e = d$ and $\epsilon_e = \epsilon/Q$. The first relation corresponds to an equivalent inflow $u_e = u/Q$, where $u$ is the inflow for the multibladed rotor.

For a two-bladed rotor, (23) and (24) yield

$$2\pi w_i(X_1) = \frac{\pi\omega k(1)e^{i\omega(1-X_1)}}{(e^{2\pi i\epsilon+2\omega d} - 1)}(1 + e^{i\psi_1+i\pi\epsilon+\omega d}) \quad (27)$$

and, if $\psi_1 = \pi$, this reduces to

$$2\pi w_i(X_1) = \pi\omega k(1)e^{i\omega(1-X_1)}/(e^{\pi i(1+\epsilon)+\omega d} - 1) \quad (28)$$

Hence, the equivalent rotor for this case has

$$d_e = d \text{ and } \epsilon_e = (1+\epsilon)/2$$

For cyclic type disturbances, where each blade has the same motion at the same point in space, $\psi_q = 2\pi q\epsilon/Q$, and

$$2\pi w_i(X_1) = \frac{\pi\omega k(1)e^{i\omega(1-X_1)}}{e^{\gamma} - 1}\left[1 + e^{\gamma}\sum_{q=1}^{Q-1} e^{-q\omega d}\right] \quad (29)$$

From the preceding analysis, it follows that Eq. (17) can be expressed in general in the form

$$2\pi W(X_1) - \frac{\pi F(Q)K(1)e^{i\nu(1-X_1)}}{QD} = -\int_{-1}^{\infty} k(X)\frac{\partial^2}{\partial Z_1^2}\left[\frac{\pi i}{2}H_0^{(2)}(\kappa r_1)\right]dX \quad (30)$$

or alternatively in terms of actual downwash and circulation as

$$2\pi w(X_1) - \frac{\pi F(Q)\,k(1)e^{i\omega(1-X_1)}}{Qd} = -\beta\int_{-1}^{\infty} k(X)e^{-i\lambda(X-X_1)}\frac{\partial^2}{\partial z_1^2}\left[\frac{\pi i}{2}H_0^{(2)}(\kappa r_1)\right]dx \quad (31)$$

In the limiting case, when $M = 0$, both equations reduce to

$$2\pi w(X_1) - \frac{\pi F(Q)k(1)e^{i\omega(1-X)}}{Qd} = -\int_{-1}^{\infty}\frac{1}{(X - X_1)}\frac{\partial k}{\partial X}dX \quad (32)$$

This equation has been solved by using the results of unsteady airfoil theory for incompressible flow.

## 4. Method of Solution

The solution of the compressible flow problem can be derived from either (30) or (31), but in the present paper the former will be used. Following the method developed by W. P. Jones,[5] Eq. (30) can be expressed in the form

$$2\pi W(X_1) + 2\pi I(X_1) - \frac{\pi F(Q)}{QD}K(1)e^{i\nu(1-X_1)} = \int_{-1}^{\infty}\frac{1}{(X_1 - X)}\frac{\partial K}{\partial X}dX \quad (33)$$

where

$$2\pi I(X_1) = \int_{-1}^{\infty}\frac{1}{X_1 - X}\frac{\partial}{\partial X}[K(X)\,S(\kappa|X - X_1|)]dX \quad (34)$$

and

$$S(\sigma) = 1 + \frac{\pi\sigma}{2}[Y_1(\sigma) + iJ_1(\sigma)] \simeq \left(\gamma - \frac{1}{2} + \log_e\frac{\sigma}{2} + \frac{i\pi}{2}\right)\left(\frac{\sigma^2}{2} - \frac{\sigma^4}{16}\right) + \frac{3\sigma^4}{64} + \text{etc.} \quad (35)$$

where $\sigma \equiv \kappa|X - X_1|$ and $\gamma$ is Euler's constant.

The problem is then reduced to one of finding a function $K(X)$ which satisfies (33) and the wake condition, $\Gamma(X) = 0$.

Let us assume that $K(X)$ can be represented to any degree of accuracy by the series

$$K(X) = U[C_0K_0(X) + C_1K_1(X) + \ldots + C_nK_n(X)] \quad (36)$$

where $K_0$, $K_1$, etc. are as defined in the list of symbols and $C_0, C_1, \ldots, C_n$ are arbitrary constants to be determined. Corresponding to the $K$ distribution we have, similarly, a $\Gamma(X)$ distribution defined by

$$\Gamma(X) = U[C_0\Gamma_0 + C_1\Gamma_1 + \ldots + C_n\Gamma_n] \quad (37)$$

All the $K_n$ functions, except $K_0$, vanish in the wake. Hence, when $X \geq 1$, we have from (12) and (36)

$$K(X) = UC_0K_0(1)e^{i\nu(1-X)} \quad (38)$$

where

$$K_0(1) = e^{-i\nu}\int_{-1}^{1}\Gamma_0 e^{i\nu X}dX = 2\pi X_0(\nu)e^{-i\nu} \quad (39)$$

with $X_0(\nu) = C(\nu)[J_0(\nu) - iJ_1(\nu)] + iJ_1(\nu)$. Further, since $X_1 = -\cos\theta_1$ and

$$e^{i\nu\cos\theta_1} = J_0(\nu) + 2\sum_{\gamma=1}^{\infty} i^r J_r(\nu)\cos r\theta_1 \quad (40)$$

it follows that, in (33)

$$K(1)e^{i\nu(1-X_1)} = 2\pi UC_0X_0(\nu)e^{-i\nu X_1} = 2\pi UC_0X_0(\nu)\left[J_0(\nu) + 2\sum_{r=1}^{\infty} i^r J_r(\nu)\cos r\theta_1\right] \quad (41)$$

Similarly, from (3) and (13),

$$W(X_1) = \frac{w'(X_1)e^{-i\lambda X_1}}{\beta} = \frac{U}{\beta}\left(\bar{\alpha} - \omega\alpha'\frac{\partial}{\partial\lambda}\right)e^{-i\lambda X_1} = \frac{U}{\beta}\left(\bar{\alpha} - \omega\alpha'\frac{\partial}{\partial\lambda}\right)\left[J_0(\lambda) + 2\sum_{r=1}^{\infty} i_r J_r(\lambda)\cos r\theta_1\right] \quad (42)$$

where $\bar{\alpha} = \alpha' + i\omega z'$. It also follows from (34) that

$$I(X_1) = U\sum_{n=0} C_nI_n, \quad (43)$$

where $I_n$, the function corresponding to $K_n$ is expressible in the form

$$I_n = \sum_{n=0} I_{nr}\cos r\theta_1 \quad (44)$$

Hence

$$I(X_1) = U\left[\sum_{n=0} C_nI_{n0} + \cos\theta_1\sum_{n=0} C_nI_{n1} + \cos r\theta_1\sum_{n=0} C_nI_{nr}\right] \quad (45)$$

The formulae for the coefficients $I_{nr}$ are given in Ref. 5, but, for particular values of $M$ and $\nu$, they may alternatively be obtained by direct numerical integration, using Fourier analysis.

**Table 1 Values of aerodynamic derivatives for a reference axis at quarter chord; $\omega = 0.1$; $d = 2.0$**

| Aero. deriv. | Mach no. | $\psi_0$ ($\equiv 2\pi\epsilon$) 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ | $5\pi/4$ | $3\pi/2$ | $7\pi/4$ |
|---|---|---|---|---|---|---|---|---|---|
| | 0.0 | 0.009 | −0.026 | 0.002 | 0.018 | 0.029 | 0.040 | 0.053 | 0.071 |
| | 0.6 | 0.014 | −0.019 | 0.022 | 0.043 | 0.057 | 0.070 | 0.085 | 0.099 |
| $l_z$ | 0.7 | 0.017 | −0.014 | 0.034 | 0.058 | 0.074 | 0.088 | 0.102 | 0.112 |
| | 0.8 | 0.019 | −0.004 | 0.058 | 0.086 | 0.102 | 0.115 | 0.127 | 0.129 |
| | 0.9 | 0.016 | 0.028 | 0.117 | 0.142 | 0.153 | 0.158 | 0.159 | 0.142 |
| | 0.0 | 0.775 | 1.692 | 1.871 | 1.892 | 1.879 | 1.845 | 1.765 | 1.504 |
| | 0.6 | 0.821 | 1.986 | 2.183 | 2.176 | 2.132 | 2.058 | 1.920 | 1.550 |
| $l_{\dot z}$ | 0.7 | 0.840 | 2.139 | 2.331 | 2.300 | 2.232 | 2.133 | 1.960 | 1.542 |
| | 0.8 | 0.866 | 2.382 | 2.531 | 2.441 | 2.326 | 2.182 | 1.958 | 1.488 |
| | 0.9 | 0.918 | 2.910 | 2.739 | 2.467 | 2.250 | 2.036 | 1.766 | 1.319 |
| | 0.0 | 0.789 | 1.671 | 1.878 | 1.914 | 1.913 | 1.890 | 1.823 | 1.580 |
| | 0.6 | 0.843 | 1.976 | 2.213 | 2.228 | 2.198 | 2.137 | 2.013 | 1.657 |
| $l_\alpha$ | 0.7 | 0.866 | 2.137 | 2.377 | 2.369 | 2.317 | 2.231 | 2.073 | 1.664 |
| | 0.8 | 0.896 | 2.396 | 2.606 | 2.543 | 2.443 | 2.312 | 2.099 | 1.629 |
| | 0.9 | 0.952 | 2.969 | 2.882 | 2.632 | 2.424 | 2.213 | 1.942 | 1.476 |
| | 0.0 | −0.149 | 4.266 | 1.659 | 0.127 | −1.010 | −2.115 | −3.503 | −5.556 |
| | 0.6 | −0.624 | 3.861 | 0.011 | −2.124 | −3.625 | −4.999 | −6.577 | −8.378 |
| $l_{\dot\alpha}$ | 0.7 | −0.825 | 3.517 | −1.106 | −3.559 | −5.219 | −6.679 | −8.256 | −9.733 |
| | 0.8 | −1.028 | 2.735 | −3.299 | −6.185 | −7.983 | −9.434 | −10.81 | −11.51 |
| | 0.9 | −0.760 | 0.073 | −9.150 | −12.00 | −13.29 | −14.02 | −14.30 | −13.07 |
| | 0.0 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| | 0.6 | 0.007 | 0.009 | 0.009 | 0.009 | 0.009 | 0.009 | 0.008 | 0.008 |
| $m_z$ | 0.7 | 0.009 | 0.012 | 0.012 | 0.011 | 0.011 | 0.011 | 0.010 | 0.009 |
| | 0.8 | 0.011 | 0.017 | 0.017 | 0.016 | 0.015 | 0.015 | 0.014 | 0.012 |
| | 0.9 | 0.017 | 0.031 | 0.026 | 0.022 | 0.020 | 0.019 | 0.017 | 0.014 |
| | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| | 0.6 | −0.006 | −0.006 | −0.011 | −0.014 | −0.016 | −0.017 | −0.018 | −0.019 |
| $m_{\dot z}$ | 0.7 | −0.011 | −0.013 | −0.025 | −0.029 | −0.032 | −0.034 | −0.036 | −0.035 |
| | 0.8 | −0.023 | −0.037 | −0.062 | −0.071 | −0.075 | −0.077 | −0.078 | −0.072 |
| | 0.9 | −0.070 | −0.172 | −0.227 | −0.233 | −0.230 | −0.224 | −0.211 | −0.178 |
| | 0.0 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 |
| | 0.6 | −0.001 | 0.001 | −0.005 | −0.008 | −0.010 | −0.011 | −0.013 | −0.014 |
| $m_\alpha$ | 0.7 | −0.006 | −0.005 | −0.017 | −0.022 | −0.025 | −0.027 | −0.029 | −0.030 |
| | 0.8 | −0.018 | −0.027 | −0.052 | −0.061 | −0.066 | −0.070 | −0.071 | −0.067 |
| | 0.9 | −0.071 | −0.162 | −0.222 | −0.231 | −0.230 | −0.225 | −0.215 | −0.184 |
| | 0.0 | −1.0 | −1.0 | −1.0 | −1.0 | −1.0 | −1.0 | −1.0 | −1.0 |
| | 0.6 | −1.355 | −1.512 | −1.527 | −1.521 | −1.512 | −1.500 | −1.479 | −1.428 |
| $m_{\dot\alpha}$ | 0.7 | −1.578 | −1.870 | −1.888 | −1.870 | =1.848 | −1.820 | −1.777 | −1.682 |
| | 0.8 | −1.986 | −2.580 | −2.567 | −2.502 | −2.440 | −2.372 | −2.275 | −2.095 |
| | 0.9 | −2.974 | −4.444 | −3.952 | −3.639 | −3.429 | −3.245 | −3.038 | −2.764 |

Finally, when the series expression for $K(X)$ is substituted in the right-hand side term of (33) we have

$$\int_{-1}^{\infty} \frac{1}{X_1 - X}\frac{\partial K}{\partial X}\,dX = 2\pi U\left[C_0 + C_1\left(\frac{1}{2} + \cos\theta_1\right) + \sum_{r=2} C_r \cos r\theta_1\right] \quad (46)$$

Comparison of the coefficients of $\cos\theta r_1$ for $r = 0, 1, 2, 3$, etc. yields the following system of simultaneous equations from which the coefficients $C_r$ can be determined:

$$C_0 + \frac{C_1}{2} = \frac{1}{\beta}\left(\bar\alpha - \omega\alpha'\frac{\partial}{\partial\lambda}\right)J_0(\lambda) + \sum_{n=0} C_n I_{n0} - \frac{\pi F(Q)}{QD}C_0 X_0(\nu)J_0(\nu)$$

$$C_r = \frac{2i^r}{\beta}\left(\bar\alpha - \omega\alpha'\frac{\partial}{\partial\lambda}\right)J_r(\lambda) + \sum_{n=0} C_n I_{nr} - 2i^r\frac{\pi F}{QD}X_0(\nu)J_r(\nu) \quad (47)$$

where $r = 1, 2, 3$, etc.

In the case of a simple oscillating airfoil without the system of wakes, the terms involving $F$ in (47) are removed, and it is known that accurate solutions can be obtained for $\omega \leq 0.5$ and $M = 0.7$ when only four terms are used in the series expression for $K$.

Only four equations of the set will then be needed in (47) to determine $C_0$, $C_1$, $C_2$, and $C_3$. Similarly, with the system of wakes present and the $F$ terms included, it is assumed that sufficient accuracy would be given by four terms only when all values of $C_r$ for $r > 3$ are assumed to be zero. For low values of $\omega$ in the range $0 < \omega < 0.2$ and $M = 0.7$ good accuracy would be obtained with even fewer terms. In the present paper, however, four terms were used to derive the values of the aerodynamic coefficients tabulated and plotted in Tables 1, 2, 3, and Figs. 3a–h, respectively.

## 5. Aerodynamic Coefficients

When the coefficients $C_0$, $C_1$, $C_2$, and $C_3$ have been determined for particular values of $Q$, $\psi_q$, $\epsilon$, $\omega$, $d$, and $M$, the $K(X)$ distribution as defined by (36) is known. The corresponding lift distribution $\bar l(X)$ is then given by (9) and (10) in the form

$$\bar l(X) = \rho U \Gamma e^{i(\lambda X + \omega T)} \quad (48)$$

**Table 2 Values of direct damping derivatives for plunging and pitching oscillations about quarter chord; $\omega = 0.2$**

| Aero deriv. | $\psi_0$ ($\equiv 2\pi\epsilon$) | $d$ = 0 | 1 | 2 | 5 | 10 | $\infty$ |
|---|---|---|---|---|---|---|---|
| $l_{\dot z}$ (M = 0.7) | 0 | ... | 0.5385 | 0.8481 | 1.2798 | 1.4974 | 1.5949 |
| | $\pi/4$ | 2.0315 | 0.7851 | 1.6693 | 1.5792 | 1.5804 | |
| | $\pi/2$ | 2.3537 | 2.2016 | 2.0731 | 1.8242 | 1.6669 | |
| | $3\pi/4$ | 2.1811 | 2.1077 | 2.0377 | 1.8635 | 1.7002 | |
| | $\pi$ | 0.9873 | 1.9422 | 1.8993 | 1.7895 | 1.6767 | |
| | $5\pi/4$ | 1.7706 | 1.7386 | 0.7114 | 1.6552 | 1.6147 | |
| | $3\pi/2$ | 1.4711 | 1.4550 | 0.4505 | 1.4762 | 1.5384 | |
| | $7\pi/4$ | 0.9229 | 0.9878 | 1.0694 | 1.2916 | 1.4843 | |
| $m_{\dot\alpha}$ (M = 0.7) | 0 | ... | −1.5255 | −1.5815 | −1.6540 | −1.6830 | −1.7025 |
| | $\pi/4$ | −1.8860 | −1.8140 | −1.7735 | −1.7255 | −1.7085 | |
| | $\pi/2$ | −1.8740 | −1.8450 | −1.8185 | −1.7630 | −1.7230 | |
| | $3\pi/4$ | −1.8090 | −1.7975 | −1.7860 | −1.7555 | −1.7240 | |
| | $\pi$ | −1.7545 | −1.7490 | −1.7440 | −1.7295 | −1.7140 | |
| | $5\pi/4$ | −1.7005 | −1.6985 | −1.6975 | −1.6965 | −1.6990 | |
| | $3\pi/2$ | −1.6330 | −1.6365 | −1.6420 | −1.6610 | −1.6845 | |
| | $7\pi/4$ | −1.5250 | −1.5505 | −1.5760 | −1.6345 | −1.6785 | |
| $l_{\dot z}$ (M = 0) | 0 | ... | 0.5049 | 0.7832 | 1.1669 | 1.3644 | 1.4552 |
| | $\pi/4$ | 1.4651 | 1.3630 | 1.3273 | 1.3480 | 1.4108 | |
| | $\pi/2$ | 1.8474 | 1.7569 | 1.6839 | 1.5522 | 1.4806 | |
| | $3\pi/4$ | 1.8831 | 1.8236 | 1.7961 | 1.6408 | 1.5266 | |
| | $\pi$ | 1.8479 | 1.7995 | 1.7545 | 1.6430 | 1.5328 | |
| | $5\pi/4$ | 1.7674 | 1.7191 | 1.6763 | 1.5807 | 1.5015 | |
| | $3\pi/2$ | 1.5985 | 1.5440 | 1.5047 | 1.4501 | 1.4426 | |
| | $7\pi/4$ | 1.1225 | 1.1122 | 1.1387 | 1.2596 | 1.3819 | |

**Table 3 Aerodynamic derivatives for a two-bladed rotor; $\omega = 0.1$; $d = 2.0$; $M = 0.7$**

| $\psi_0$ | $\psi_1$ | $l_z$ | $l_{\dot z}$ | $l_\alpha$ | $l_{\dot\alpha}$ | $m_z$ | $m_{\dot z}$ | $m_\alpha$ | $m_{\dot\alpha}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.0166 | 0.840 | 0.8657 | −0.825 | 0.0086 | −0.0110 | −0.0058 | −1.578 |
| | $\pi/2$ | 0.0658 | 0.872 | 0.9464 | −5.729 | 0.0084 | −0.0210 | −0.0164 | −1.561 |
| | $\pi$ | 0.0742 | 2.232 | 2.3169 | −5.219 | 0.0111 | −0.0320 | −0.0246 | −1.848 |
| | $3\pi/2$ | −0.0268 | 1.154 | 1.1372 | 3.827 | 0.0096 | −0.0040 | 0.0020 | −1.665 |
| $\pi/4$ | 0 | −0.0393 | 1.597 | 1.5683 | 5.521 | 0.0106 | −0.0040 | 0.0026 | −1.766 |
| | $\pi/2$ | 0.0226 | 1.521 | 1.5538 | −0.729 | 0.0100 | −0.0170 | −0.0103 | −1.720 |
| | $\pi$ | 0.0811 | 2.187 | 2.2794 | −5.951 | 0.0110 | −0.0330 | −0.0259 | −1.835 |
| | $3\pi/2$ | −0.0712 | 2.677 | 2.6185 | 9.787 | 0.0130 | −0.0050 | 0.0043 | −2.011 |
| $\pi/2$ | 0 | −0.0139 | 2.139 | 2.1368 | 3.517 | 0.0115 | −0.0130 | −0.0054 | −1.870 |
| | $\pi/2$ | 0.0341 | 1.820 | 1.8647 | −1.615 | 0.0105 | −0.0210 | −0.0142 | −1.779 |
| | $\pi$ | 0.0878 | 2.133 | 2.2312 | −6.679 | 0.0108 | −0.0340 | −0.0271 | −1.820 |
| | $3\pi/2$ | 0.0308 | 2.949 | 2.9924 | −0.159 | 0.0129 | −0.0280 | −0.0188 | −2.021 |
| $3\pi/4$ | 0 | 0.0145 | 2.299 | 2.3249 | 0.825 | 0.0117 | −0.020 | −0.0122 | −1.890 |
| | $\pi/2$ | 0.0469 | 1.904 | 1.9614 | −2.809 | 0.0106 | −0.024 | −0.0173 | −1.791 |
| | $\pi$ | 0.0946 | 2.061 | 2.1663 | −7.438 | 0.0106 | −0.035 | −0.0282 | −1.801 |
| | $3\pi/2$ | 0.0729 | 2.761 | 2.8460 | −4.560 | 0.0122 | −0.035 | −0.0269 | −1.961 |
| $\pi$ | 0 | 0.0341 | 2.331 | 2.3767 | −1.106 | 0.0116 | −0.025 | −0.0166 | −1.888 |
| | $\pi/2$ | 0.0564 | 1.896 | 1.9635 | −3.374 | 0.0105 | −0.026 | −0.0193 | −1.784 |
| | $\pi$ | 0.1018 | 1.960 | 2.0725 | −8.256 | 0.0104 | −0.036 | −0.0293 | −1.777 |
| | $3\pi/2$ | 0.0972 | 2.602 | 2.7114 | −7.155 | 0.0117 | −0.039 | −0.0313 | −1.916 |
| $5\pi/4$ | 0 | 0.0479 | 2.323 | 2.3823 | −2.492 | 0.0115 | −0.0270 | −0.0195 | −1.879 |
| | $\pi/2$ | 0.0628 | 1.827 | 1.9001 | −4.483 | 0.0104 | −0.0270 | −0.0203 | −1.767 |
| | $\pi$ | 0.1088 | 1.805 | 1.9238 | −9.112 | 0.0100 | −0.0360 | −0.0300 | −1.740 |
| | $3\pi/2$ | 0.1200 | 2.445 | 2.5762 | −9.594 | 0.0112 | −0.0430 | −0.0354 | −1.871 |
| $3\pi/2$ | 0 | 0.0583 | 2.300 | 2.3693 | −3.559 | 0.0114 | −0.029 | −0.0216 | −1.870 |
| | $\pi/2$ | 0.0644 | 1.682 | 1.7561 | −4.790 | 0.0100 | −0.026 | −0.0200 | −1.735 |
| | $\pi$ | 0.1124 | 1.542 | 1.6636 | −9.733 | 0.0094 | −0.035 | −0.0295 | −1.682 |
| | $3\pi/2$ | 0.1486 | 2.179 | 2.3377 | −12.72 | 0.0105 | −0.047 | −0.0403 | −1.801 |
| $7\pi/4$ | 0 | 0.0668 | 2.269 | 2.3467 | −4.44 | 0.0112 | −0.031 | −0.0232 | −1.859 |
| | $\pi/2$ | 0.0516 | 1.407 | 1.4679 | −3.774 | 0.0096 | −0.022 | −0.0159 | −1.682 |
| | $\pi$ | 0.0959 | 1.099 | 1.2033 | −8.516 | 0.0086 | −0.029 | −0.0239 | −1.595 |
| | $3\pi/2$ | 0.1677 | 1.451 | 1.6279 | −15.36 | 0.0089 | −0.046 | −0.0409 | −1.637 |
| $2\pi$ | 0 | 0.0742 | 2.232 | 2.3169 | −5.219 | 0.0111 | −0.032 | −0.0246 | −1.848 |
| | $\pi/2$ | −0.0268 | 1.154 | 1.1372 | 3.827 | 0.0096 | −0.0040 | 0.0020 | −1.665 |
| | $\pi$ | 0.0166 | 0.840 | 0.8657 | −0.825 | 0.0086 | −0.0110 | −0.0058 | −1.578 |
| | $3\pi/2$ | 0.0658 | 0.872 | 0.9464 | −5.729 | 0.0084 | −0.0210 | −0.0164 | −1.561 |

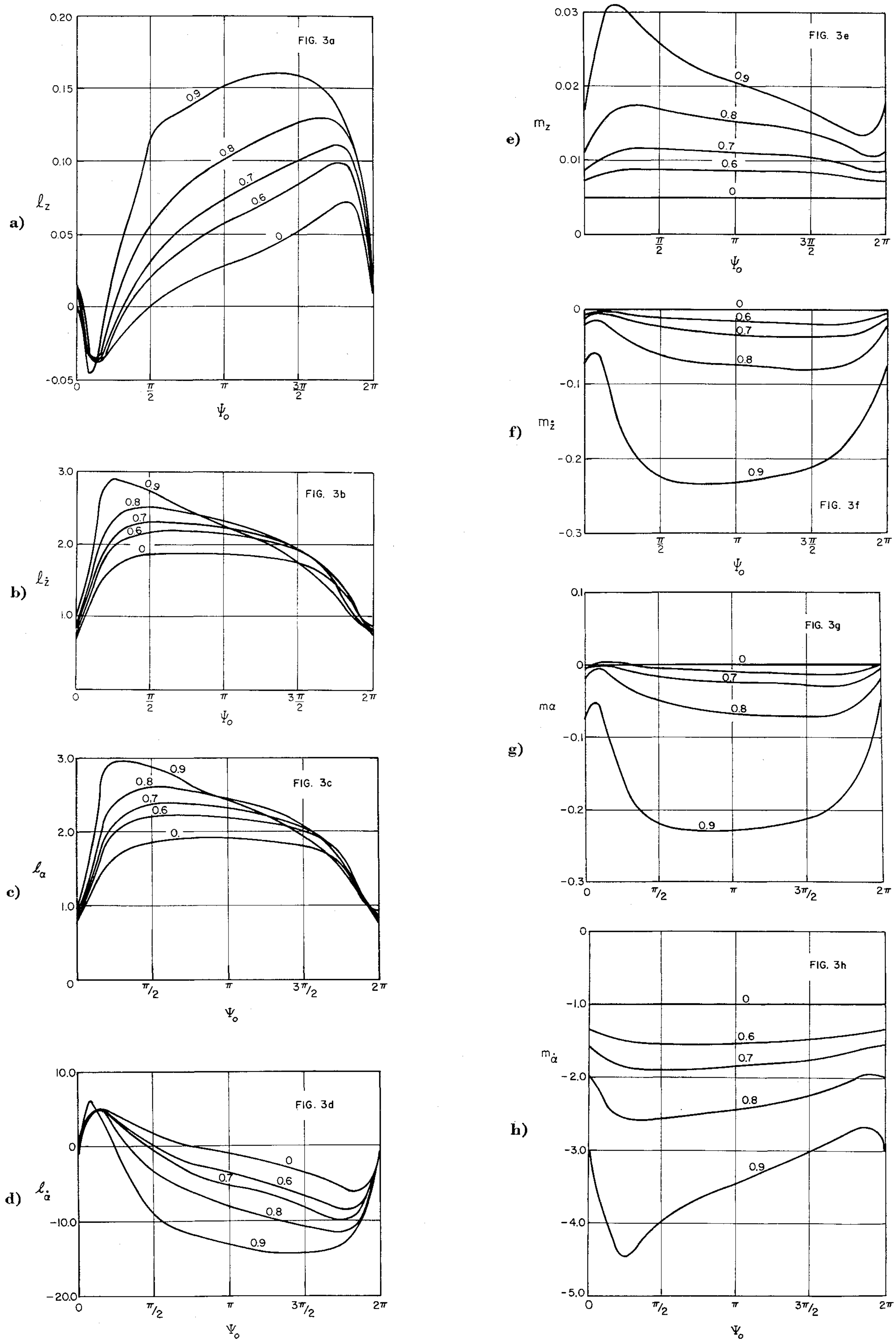


Fig. 3 Influence of Mach number and phase on aerodynamic coefficients ($\omega = 0.1$; $d = 2.0$).

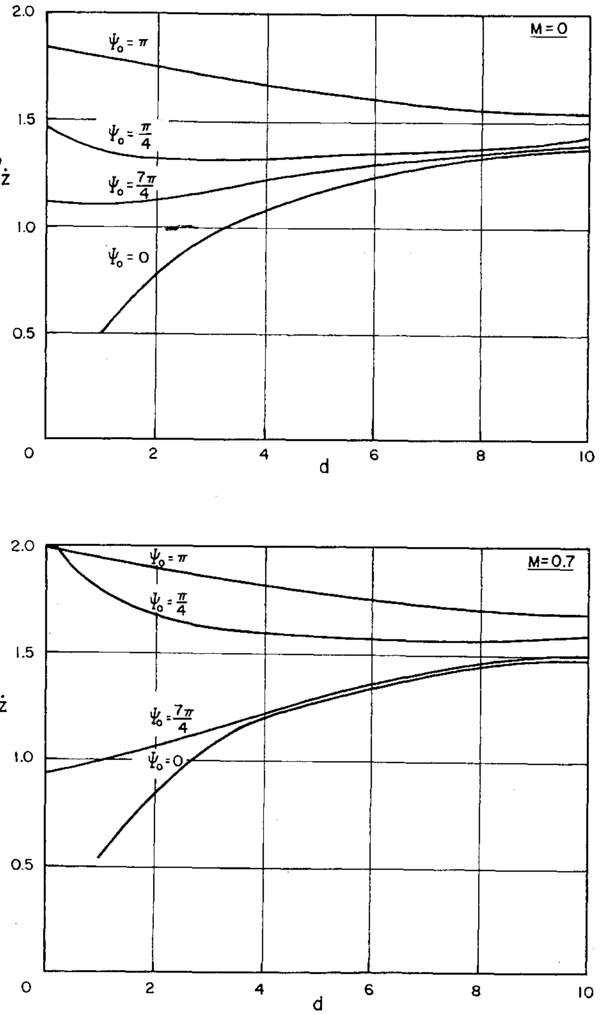


**Fig. 4 Variation in $l_{\dot z}$ for $M = 0$ and $M = 0.7$ when $\omega = 0.2$.**

It also follows that the total lift $L$ would be given by

$$L = \rho l U^2 \sum_{n=0} C_n R_n e^{i\omega T}$$

where

$$R_n(\lambda) = \int_{-1}^{1} \Gamma_n e^{i\lambda X} dX \qquad (49)$$

Similarly, it can be deduced that the nose-up pitching moment $M$ about the midchord point is

$$M = i\frac{\partial L}{\partial \lambda} = i\rho l^2 U^2 \sum_{n=0} C_n R_n' e^{i\omega T} \qquad (50)$$

where

$$R_n'(\lambda) = \partial R_n(\lambda)/\partial\lambda$$

The formulae for $R_n$ and $R_n'$ are given in the list of symbols and the $C_n$ coefficients are linear functions of $z'$ and $\alpha'$ determined by (47). It then follows that $L$ and $M$ can be expressed in terms of the displacements $z$ and $\alpha$ in the form

$$\frac{L}{\pi \rho l U^2} = (l_z + i\omega l_{\dot z})z + (l_\alpha + i\omega l_{\dot\alpha})\alpha$$
$$\frac{M}{\pi \rho l^2 U^2} = (m_z + i m_{\dot z})z + (m_\alpha + i\omega m_{\dot\alpha})\alpha \qquad (51)$$

where $l_z$, $l_{\dot z}$, etc. are aerodynamic coefficients.

Numerical values of the aerodynamic coefficients referred to the midchord axis were obtained for a range of values of $\omega$ and $M$. By using the standard transformation formulae, the appropriate aerodynamic coefficients for a reference axis at quarter chord were also derived and these are shown plotted and tabulated in this paper. The calculations were done using the IBM 360/65 computer of Texas A&M University's Data Processing Center.

## 6. Discussion of Results

As stated earlier in this report, the downwash $w_i(x_1)$ due to the infinite system of wakes with given circulation is independent of Mach number and is proportional to a complex function $F(Q)$ which is shown plotted in Figs. 2a and b. For given $d$ values of the spacing between the vortex sheets it is evident from the diagram that $|F|$ is greatest when $\omega$ is small and when $\psi_0$ is near zero (or $2\pi$). Hence one would expect greatest interference from the wakes when $\psi_0 \to 0$. This has already been established for incompressible flow and according to the present analysis is also the case for compressible flow.

The problem considered bears some resemblance to that of determining the wind-tunnel interference effects of a system of image wakes but in that case the wakes only extend downstream from the image airfoils and not from $-\infty$ to $\infty$.[7] It has been proved theoretically and established experimentally by Woolston and Runyan[8] that "resonance" can occur in the wind tunnel at certain critical frequencies defined by

$$f_c = p/2\pi = [(2m - 1)U_s(1 - M^2)^{1/2}/2dl]$$

where $dl$ is the tunnel height and $m$ is an integer. In the present problem, such resonances do not occur and there appear to be no significant effects of this type.

For a rotating single blade the influence of its system of wakes is greatest when $\psi_0 = 0$, that is when $p/\Omega$ is an integer. As far as one can judge from the figures and tables given in this report there are no unexpected changes in the aerodynamic coefficients with a change in Mach number. Figures 3a–h show the aerodynamic coefficients $l_z$, $l_{\dot z}$, etc., plotted against $\psi_0$ for different Mach numbers for $\omega = 0.1$ and $d = 2.0$. The curves of $l_{\dot z}$ and $l_{\dot\alpha}$ for $M = 0$ are identical with those given by J. P. Jones[2] for incompressible flow.

All the aerodynamic coefficients vary with Mach numbers but the variation with respect to $\psi_0$ follows the pattern for the case $M = 0$. The direct damping coefficient for plunging oscillations, $l_{\dot z}$, and for pitching oscillation about quarter-chord, $-m_{\dot\alpha}$, are both positive and increase as $M$ is increased, as shown in Figs. 3b and h.

Figure 4 compares the values of $l_{\dot z}$ for $M = 0$ and $M = 0.7$ for different values of $d$ when $\omega = 0.2$ and $\psi_0$ is varied. In both cases the damping is much lower when $\psi_0 = 0$ than when $\psi_0 = \pi$ but never becomes negative. The pitching moment damping coefficient, $-m_{\dot\alpha}$, is constant for $M = 0$ and varies little with $\psi_0$ when $M = 0.7$ as shown in Table 2.

The aerodynamic coefficients for a two-bladed rotor, when one blade has a lead in phase of $\psi_1$, are given in Table 3 for the particular case when $\omega = 0.1$, $d = 2.0$ and $M = 0.7$ for different values of $\psi_0$ and $\psi_1$. For all cases considered, $l_{\dot z}$ and $-m_{\dot\alpha}$ are positive and do not vary greatly, so that any plunging or pitching oscillation about the quarter-chord axis would be damped. The aerodynamic coefficient $l_{\dot\alpha}$ however, shows considerable variation with $\psi_1$ and could be an important factor in determining the critical speed for flutter.

For oscillations about axis positions forward of quarter-chord, it is possible to get one degree of freedom instability since for low values of $\omega$ the pitching damping coefficient, $-m_{\dot\alpha}$, can then be negative.

The few results given in this paper serve to illustrate that the method used to calculate the aerodynamic coefficients is satisfactory, but it is realized that in a flutter analysis of a multibladed rotor much more data would be required. If two-dimensional strip theory is used in such an analysis, aerodynamic coefficients would be required for a range of $\omega$ and $d$ values. Further calculations are planned with the aim of producing tables of aerodynamic coefficients for a two-bladed rotor in hovering flight for a range of practical values of $\omega$, $d$, $\psi_0$, $\psi_1$, and $M$.

### $\Gamma_n$, $K_n$ Distributions

$$\Gamma_n = i\nu K_n + \frac{\partial K_n}{\partial X}, \qquad K_n = e^{-i\nu x}\int_{-1}^{x} \Gamma_n\, e^{i\nu x} dX$$

$$\Gamma_0 = 2[C(\nu)\cot(\theta/2) + i\nu \sin\theta]$$

$$\Gamma_1 = -2\sin\theta + \cot(\theta/2) + i\nu\,[\sin\theta + \sin 2\theta/2]$$

$$\Gamma_n = -2\sin n\theta + i\nu\{[\sin(n+1)\theta/(n+1)] - [\sin(n-1)\theta/(n-1)]\};\ n \geq 2$$

$$K_0 = 2\pi X_0(\nu)e^{-i\nu x},\ X \geq 1$$

$$K_1 = \sin\theta + \sin 2\theta/2$$

$$K_n = \sin(n+1)\theta/(n+1) - \sin(n-1)\theta/(n-1) \quad n \geq 2$$

$$C(\nu) = H^{(2)}(\nu)/(H_1^{(2)}[(\nu) + iH_0^{(2)}(\nu)$$

$$X_0(\nu) = C(\nu)J_0(\nu) + i[1 - C(\nu)]J_1(\nu)$$

$J_0, J_1$ = Bessel functions

$H_0^{(2)}, H_1^{(2)}$ = Hankel functions

### Lift and Moment Integrals

$$R_0 = 2\pi\{C(\nu)[J_0(\lambda) - iJ_1(\lambda)] + i\nu/2[J_0(\lambda) + J_2(\lambda)]\}$$

$$R_1 = -\pi(1 - \nu/\lambda)\ [J_2(\lambda) + iJ_1(\lambda)]$$

$$R_n = (-i)^{n+1}\pi[1 - (\nu/\lambda)][J_{n+1}(\lambda) + J_{n-1}(\lambda)];\ n \geq 2$$

## References


[1] Loewy, R. G., "A Two-Dimensional Approximation to the Unsteady Aerodynamics of Rotary Wings," *Journal of the Aeronautical Sciences*, Vol. 24, No. 2, Feb. 1957, pp. 81–92.

[2] Jones, J. P., "The Influence of the Wake on the Flutter and Vibration of Rotor Blades," *Aeronautical Quarterly*, Vol. IX, 1958, pp. 258–286.

[3] Timman, R. and Van de Vorren, A. I., "Flutter of a Helicopter Rotor Rotating in Its Own Wake," *Journal of the Aeronautical Sciences*, Vol. 24, No. 9, Sept. 1957, pp. 694–702.

[4] Greidanus, J. H., Timman, R., and Zaat, J. A., "The Aerodynamic Forces on Oscillating Blades of a Helicopter Rotor with Finite Span, Rotating in Still Air," N.L.L. Rept. F 108, 1952, National Lucht-En Ruemtevaarlaboratorium, Amsterdam.

[5] Jones, W. P., "The Oscillating Airfoil in Subsonic Flow," R&M 2921, 1956, Aeronautical Research Council.

[6] Infeld, L., Smith, V. G., and Chien, W. Z., "On Some Series of Bessel Functions," *Journal of Mathematics and Physics*, April 1947, pp. 22–28.

[7] Jones, W. P., "Wind-Tunnel Wall Interference Effects on Oscillating Airfoils in Subsonic Flow," R&M 2943, 1957, Aeronautical Research Council.

[8] Woolston, D. S. and Runyan, H. L., "Some Considerations on the Air Forces on a Wing Oscillating Between Two Walls for Subsonic Compressible Flow," *Journal of the Aeronautical Sciences*, Vol. 22, Jan. 1955, pp. 41–50.